THE LIMIT OF TRANSITIVITY OF A SUBSTITUTION GROUP*

BY

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1. Jordan published his final contribution to the problem of the limit of transitivity of a substitution group which does not contain the alternating group in Liouville's Journal of 1895.‡ Here he gave the interesting inequalities between n, the degree, and t, the multiplicity of transitivity, of a t-ply transitive group, stated in the following theorem:

Let n be the degree of a t-ply transitive group G of class >3. Then if $t \geqq 8$,

$n-t \geqq 2^{\alpha}$, *α an integer $\geqq k-3-\log k/\log 2$, and k an integer such that $5 \leqq k \leqq t$; or*

$n-t \geqq t!/\{\delta!(t-\delta)!\}$, *$\delta$ being the greatest integer less than the quantity $t-(t-k+1)\log 2/(k+\log 2)$.*

This paper followed Bochert's study of the problem in the Mathematische Annalen of 1887 and 1889. Among the results of this study was the inequality

$$\log n \geqq a(t \log t)^{1/2},$$

where the constant a may be taken to be $(1/8)\{(\log 2)/8\}^{1/2}$, if $t \geqq 8$.§

In recent years Miller has given still another relation between n and t, namely, $n \geqq (4/25)(t+2)^2$.||

In this paper inequalities similar to those of Jordan will be established. To obtain these it was necessary to study separately the t-ply transitive group whose subgroup that fixes t letters is of order a power of two and the t-ply transitive group whose subgroup that fixes t letters has its order divisible by an odd prime. The methods used in the study of the former group, though suggested by Jordan's 1895 paper, are new, while those used in the investigation of the latter group follow to a certain extent those of Jordan. The following theorems give the chief results of this study:

Let the subgroup that fixes t letters of a t-ply transitive group of degree n and class >3 be of order 2^m. Then if $t \geqq 8$,

$$(n-t)/2 \geqq t!/\{\beta!(t-\beta)!\},$$

* Presented to the Society, August 29, 1929; received by the editors in August, 1929.

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‡ C. Jordan, Journal de Mathématiques, (5), vol. 1 (1895), pp. 35–60.

§ A. Bochert, Mathematische Annalen, vol. 33 (1889), p. 577.

|| G. A. Miller, Proceedings of the National Academy of Sciences, vol. 2 (1916), p. 61.

where β is an integer chosen such that $t/2<\beta\leqq t-3$, or chosen to be one unit less than the least value of β which satisfies the inequality

$$[(\beta+1)/2]! \geqq 2t!/\{\beta!(t-\beta)!\};$$

or

$$n-t \geqq [(t+1)/2]!;$$

or

$$n-t \geqq t!/[t/2]!^2.$$

The symbol $[s]$ denotes the integral part of s.

Let the order of the subgroup that fixes t letters of a t-ply transitive group of degree n and class >3 be divisible by an odd prime. Then if $t\geqq 8$,

$$(n-2t+1)/2 \geqq 2^\alpha, \ \alpha \text{ an integer} > 3 \text{ and } \geqq t-3-\log t/\log 2;$$

or

$$(n-2t+1)/2 \geqq 2^\gamma t, \ \gamma \text{ an integer} > 3 \text{ and } \geqq k-3-\log k/\log 2,$$

where k is an integer such that $5\leqq k\leqq t-1$; or

$$(n-2t+1)/2 \geqq t!/\{\delta!(t-\delta)!\},$$

where δ is the greatest integer less than the quantity $t-(t-k+1)\log 2/(k+\log 2)$.

It may be of interest to see how Jordan's, Miller's, and the author's results compare for a few values of t. In the following table the minimum value of n has been calculated for a given t from the above formulas:

t	8	16	25	50	100
Jordan	16	32	50	1,275	161,800
Miller	16	52	117	433	1,665
Author	31	271	849	6,499	409,799

Bochert's inequality is of no interest for these small values of t.

2. The first step in the study of the t-ply transitive group G is the proof of the following lemma:

LEMMA. *If $t>5$, the subgroup that fixes t letters of a t-ply transitive group fixes exactly t, $t+1$, or $t+2$ letters.*

Let G_x be the subgroup that fixes x letters of a t-ply transitive group G. Suppose that G_t fixes the $t+r-1$ letters $b_1, b_2, \cdots, b_t, a_1, a_2, \cdots, a_{r-1}$, $r>1$. Now consider the largest subgroup of G_{t-1} in which G_t is invariant. Its order is rg_t, where g_t is the order of G_t, and it has a regular constituent of degree r on the letters $b_1, a_1, a_2, \cdots, a_{r-1}$. Since all the subgroups similar to G_t found in G_{t-1} are conjugate, all the subgroups similar to G_t in any of the preceding groups G_{t-i}, $i=2, 3, \cdots, t$, $G_0=G$, form one complete set of conjugates. Then the largest subgroup of G_{t-2} in which G_t is invariant has a doubly transitive constituent of degree $r+1$ on the letters $b_2, b_1, a_1, a_2, \cdots,$

a_{r-1}.* Finally, the largest subgroup of G in which G_t is invariant has a t-ply transitive constituent of degree $r+t-1$ and order $r(r+1)\cdots(r+t-1)$ on the letters that G_t fixes. Now Jordan has shown that such a non-alternating t-ply transitive group does not exist unless $t\leqq 3$, or $t=4$ and $r=8$, or $t=5$ and $r=8$.† Hence if $t>5$, the t-ply transitive constituent is alternating or symmetric and $r=2$, or 3.

In the remaining part of the paper it will be assumed that $t>5$. The study of the problem will be divided into two parts. In the first part the special case G_t of order 2^m will be studied, while in the second part the more general case G_t of order $p^\alpha q$, p an odd prime, will be investigated. It will be found that the former case leads to relations between n and t which give more favorable results than those derived from the more general case. Thus when t is sufficiently large, the former relations may be neglected.

G_t OF ORDER 2^m

3. Suppose that G_t is of order 2^m and that it contains a transitive subgroup. Let M of degree 2^γ be its largest transitive subgroup. Now G_t cannot have two transitive subgroups on different sets of letters, for $2^\gamma\geqq u$, the class of G, and $u\geqq(n-1)/2$.‡ Then let G_{t-1} be of degree $2^\gamma+q$. From the theory of primitive groups with transitive subgroups of lower degree,§ it is known that q divides 2^γ, and that G_{t-1} is imprimitive with systems of imprimitivity of q letters. Hence G_t fixes exactly $t+1$ letters, for q would be odd in case G_t fixed t or $t+2$ letters, the only other possibilities by the above lemma. Now let the letters displaced by G_{t-1} but fixed by M be $a_1', a_2', \cdots, a_q'$; the letters displaced by G_{t-1} but fixed by G_t be a_1' and a_2'; the letters introduced by $G_{t-2}, G_{t-3}, \cdots, G$, respectively, be $b, b_1, \cdots, b_{t-2}$. Then the letters $a_1', a_2', \cdots, a_q'$ form one system of imprimitivity of G_{t-1}, and let $a_1, a_2, \cdots, a_q$ be the letters of another system of imprimitivity of G_{t-1}.

Now consider the largest group I of G in which G_t is invariant. It transforms M into itself, for M is the largest transitive subgroup of G_t. If I did not transform M into itself, it would transform M into a conjugate M' which, as has been seen in the above paragraph, is not entirely free from

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* W. A. Manning, Bulletin of the American Mathematical Society, vol. 13 (1906), p. 20.

† C. Jordan, Journal de Mathématiques, (2), vol. 17 (1872), p. 351.

‡ W. A. Manning, these Transactions, vol. 31 (1929), p. 648.

§ C. Jordan, Journal de Mathématiques, (2), vol. 16 (1871), pp. 383–408.

Marggraff, Dissertation, *Ueber primitive Gruppen mit transitiven Untergruppen geringeren Grades*, Giessen, 1889.

W. A. Manning, these Transactions, vol. 7 (1906), pp. 499–508.

W. A. Manning, *Primitive Groups*, 1921, Chap. VI. The reader is referred to this last reference for a brief yet complete account of the theory.

the letters of M. The transitive groups M and M' then would generate a transitive subgroup of degree or order greater than M.

Further I has a symmetric constituent on the letters a_1', a_2', b, $b_1, \cdots$, b_{t-2}. Hence it contains the substitution

$$T = (bb_1b_2)(a_1')(a_2')(b_3)(b_4) \cdots (b_{t-2}) \cdots .$$

It may be assumed that T fixes a_1, for if T displaces the letter a_1, the product of T and a properly chosen substitution from M gives a substitution fixing a_1. Since the set of letters a_1', a_2', $\cdots$, a_q' and the set of letters $a_1, a_2, \cdots$, a_q are the letters of two systems of imprimitivity of G_{t-1}, T permutes the letters of each of these sets among themselves. Proper powers of T will reduce it to a substitution of order three. It will be shown that this substitution occurs in a conjugate of G_t. Now it cannot displace all the letters $a_2, a_3, \cdots$, a_q and all the letters a_3', a_4', $\cdots$, a_q', for then $q-2\equiv 0$, mod 3, and $q-1\equiv 0$, mod 3. Thus this substitution fixes another letter a_x. Hence it occurs in the G_t which fixes $b_3, b_4, \cdots, b_{t-2}, a_1', a_2', a_1$, and a_x. The following theorem has then been proved:

THEOREM 1. *If $t>5$, and if the subgroup which fixes t letters of a t-ply transitive group G is of order 2^m, G contains no transitive subgroup of degree $<n-t+1$.*

4. Let I' be that subgroup of G, in which G_t is invariant, which has an alternating constituent on the t letters $b_1, b_2, \cdots, b_t$ that G_t fixes. Let a's be the letters that G_t displaces. Further assume that I' permutes among themselves the letters of each transitive constituent of G_t. Then each substitution $(b_1b_2b_3)(b_4)(b_5) \cdots (b_t)(a \cdots) \cdots$ of order three of I' fixes at least one letter of each transitive constituent of G_t, for each such transitive constituent is of degree a power of two. Hence G_t has exactly two transitive constituents, for if it had three or more, these substitutions would occur in the G_t which fixed the three or more a's of the transitive constituents fixed and the letters $b_4, b_5, \cdots, b_t$. The case when G_t has only one transitive constituent falls under the previous discussion (§3). Moreover G_t fixes exactly $t(>5)$ letters, for if it fixed more, the substitution $(b_1b_2b_3)(b_4)(b_5) \cdots (b_t)(a_u)(a_v) \cdots$ of order three would occur in the G_t which fixed the letters $b_4, b_5, \cdots, b_t, a_u, a_v, x$, where a_u and a_v are the letters of G_t that such a substitution fixes and x is the additional letter fixed by G_t.*

If G_{t-1} is imprimitive it has systems of imprimitivity of degree $2^\gamma+1$,

* If $t=5$; G_t (see lemma, §2) may fix 5, 6, 7, or 12 letters. However in the last three cases the substitution $(b_1b_2b_3)(b_4) \cdots (b_t)(a_u)(a_v) \cdots$ of order three again fixes more than t letters. Thus G_t also fixes exactly t letters if $t=5$.

where 2^γ is the degree of one of the transitive constituents of G_t. Let the other transitive constituent of G_t be of degree 2^δ. Then $2^\gamma+1$ divides $2^\gamma+1+2^\delta$, the degree of G_{t-1}. This division is obviously impossible, and hence G_{t-1} is primitive.

Now consider the subgroup F of G_t which fixes one letter of one of the transitive constituents of G_t. If F reduces to the identity, G_t has a regular constituent simply or multiply isomorphic to the second transitive constituent. If there is a multiple isomorphism between the constituents, let F be the subgroup which fixes one letter from the constituent of smaller order. If the transitive constituents are in simple isomorphism, each constituent is regular or one is not. The former case will be considered later. In the latter case take F to be the subgroup which fixes one letter of the non-regular constituent. Thus, unless G_t is a simple isomorphism between two regular constituents, a subgroup F of G_t may be found which does not reduce to the identity.

Now let X and Y be the two transitive constituents of G_t. Further say that the letter a_1 of G_t that F fixes is found in the constituent X. Then F fixes $2^\beta>1$ letters of X. Assume that F fixes none of the letters of Y. In G_t the largest group N in which F is invariant has a regular constituent on the 2^β letters fixed by F. Since G_{t-1} is primitive and since G_t has exactly two transitive constituents, G_{t-1} contains a substitution* $S=(b_1a_1)\cdots$ which transforms F into itself and hence permutes among themselves the remaining $2^\beta-1$ letters of X that F fixes. Consequently the group $\{N, S\}$ has a doubly transitive constituent of order $2^\beta(2^\beta+1)$ on the 2^β letters of X and the letter b_1, for the subgroup that fixes b_1 of $\{N, S\}$ is in G_t and hence is N.

The G_{t-1} which fixes b_2 but displaces b_1 likewise contains a substitution $S'=(b_2a_1)\cdots$ which transforms F into itself. The group $\{N, S, S'\}$ has a 3-ply transitive constituent of order $2^\beta(2^\beta+1)(2^\beta+2)$. Thus in the t-ply transitive group G a group $N'=\{N, S, S', S'', \cdots\}$ can be found which has a $(t+1)$-ply transitive constituent of degree $2^\beta+t$ and of order $2^\beta(2^\beta+1)(2^\beta+2)\cdots(2^\beta+t)$. However, Jordan has shown that such a multiply transitive constituent is alternating or symmetric if $t>4$. Thus $2^\beta=2$, and the multiply transitive constituent of N' is symmetric.

Now consider a substitution $(b_1b_2b_3)(b_4)(b_5)\cdots(b_t)(a_1)(a_2)\cdots$ of order three of N', a_1 and a_2 being the two letters fixed by F. Since this substitution also transforms G_t into itself, it must also fix at least one letter, a_x, of the constituent Y of G_t. Thus it occurs in the G_t which fixes $a_1, a_2, a_x, b_4, b_5, \cdots,$ b_t.

* W. A. Manning, these Transactions, vol. 29 (1927), p. 815, §§1 and 5, Corollary II.

Hence F fixes some letters from each transitive constituent of G_t. First, it will be shown that the transitive constituents of G_t are simply isomorphic. Suppose them multiply isomorphic. Choose F as the subgroup which fixes one letter of the constituent of smaller order. Then, contrary to the above statement, F does not fix letters from each transitive constituent of G_t.

Further it will be shown that the two constituents are of the same degree. Let F be the subgroup which fixes one letter of the constituent of smaller degree. The order of this constituent will be wg_1, where w is the degree of the constituent and g_1 is the order of F. Since F also fixes letters of the second constituent of G_t, the order g_2 of the subgroup that fixes one letter of this constituent is at least g_1. Let m be the degree of the second constituent. Then $wg_1 = mg_2$, where $g_2 \geqq g_1$ and $m \geqq w$. Hence $g_1 = g_2$ and $w = m$.

Thus F fixes 2^β letters from each transitive constituent of G_t and N has two simply isomorphic regular constituents on the letters which F fixes. If none of the substitutions $S, S', \cdots$ connect the letters of the two regular constituents of N, the group N' may be formed and the reasoning of the above paragraphs may be applied to its multiply transitive constituent. As before $2^\beta = 2$ and in this case the substitution $(b_1b_2b_3)(a \cdots) \cdots$ of order three of N' fixes $b_4, b_5, \cdots, b_t$ and the 4 letters which F fixes. Hence it occurs in a conjugate of G_t. Then at least one S connects the sets of letters in question. Assume this substitution to be S. Then the group $\{N, S\}$ has a primitive constituent of degree $2^{\beta+1}+1$ and of class $2^{\beta+1}$ on the letters that F fixes and the letter b_1. Hence this constituent is of degree p^α, p a prime, and contains a characteristic elementary subgroup.*

As before, form the group N'. Since the substitution $(b_1b_2b_3)(b_4)(b_5) \cdots (b_t)(a_1) \ldots$ of order three of N' transforms both G_t and F into itself, it permutes among themselves the letters of each set of 2^β a's that F fixes. Hence it fixes exactly one a from each of these sets, for if it fixed more than one letter from each set, this substitution would occur in a conjugate of G_t. Thus $2^\beta \equiv 1$, mod 3, and $p = 3$. The group N' cannot have so much as a 5-ply transitive constituent in the letters in question, for let $T = (a_1a_2)(a_3a_4) \cdots$ be a substitution of order two of N on these letters. Then in N' there exist the substitutions

$$V = \begin{pmatrix} b_1a_1a_2a_3a_4b_6 \cdots \\ b_1b_2b_3b_4b_5b_6 \cdots \end{pmatrix} \text{ and } V^{-1}TV = (b_1)(b_2b_3)(b_4b_5)(a_xa_y) \cdots .$$

Note that the substitution $V^{-1}TV$ transforms the elementary group of degree 3^α into itself. Now the partial substitution $U = (a_xa_y) \cdots$ fixes 5 letters of

* G. Frobenius, Berliner Sitzungsberichte, 1902, pp. 455–459.

the constituent of N in question and transforms the elementary subgroup of $\{S, N\}$ of degree 3^α into itself. This is obviously impossible. Hence $t \leqq 4$, contrary to the present hypothesis.

Then the subgroup F reduces to the identity and G_t is of class $n-t$. Since G_{t-1} is primitive, it is of degree p^α, p a prime, and contains a characteristic elementary subgroup. Further since G_t is of class $n-t$, each of its transitive constituents is of degree 2^γ. Now $2^\gamma \equiv 1$, mod 3, for the substitution $(b_1b_2b_3)(b_4) \cdots (b_t) \cdots$ of order three of I' fixes only one a from each of the transitive constituents of G_t. Then $2^\gamma + 2^\gamma + 1 \equiv 0$, mod 3, and $p=3$. Apply the reasoning of the previous paragraph to the groups G, G_{t-1}, and G_t, and it is seen that $t \leqq 4$.

Then since $t>5$, I' permutes some, and hence t or more, transitive constituents of G_t. Thus the following theorem has been proved:

THEOREM 2. *If G_t, the subgroup that fixes t letters of a t-ply transitive group G, is of order 2^m, the largest subgroup in which G_t is invariant permutes $t(t>5)$ or more transitive constituents of G_t.**

5. At this point it is necessary to assume $t \geqq 8$, and this assumption will be held throughout the remaining part of this paper. It has been shown that the substitutions of I' permute some transitive constituents of G_t. Hence I' contains an imprimitive constituent, whose systems of imprimitivity are transitive constituents of G_t and which is multiply isomorphic to the alternating group of degree t. Consider the group H whose letters are the systems of imprimitivity of the imprimitive constituent just described. It is simply isomorphic to the alternating constituent A of degree t in I'. Let A_1 be the subgroup of A which is simply isomorphic to H_1, the subgroup which fixes one letter of H. Now A_1 has either one transitive constituent of degree $> \frac{1}{2}t$, or all of its transitive constituents are of degree $\leqq \frac{1}{2}t$. Let the former be true, and call the transitive constituent of degree $\beta > \frac{1}{2}t$, B. Let B contain the alternating group of degree β of A_1 which fixes $t-\beta$ letters of A.

Under this hypothesis, it will be shown that A_1 cannot be transitive. Note that if it is transitive it is alternating. Let A_1 be transitive and of degree $t-1$. Then H is of degree t and is the alternating group. Thus a substitution of degree and order three of A_1 is associated† with a substitution of degree and order three of H_1. Since $t \geqq 8$, this substitution fixes more than three letters of H and in I' this substitution which has a cycle of three letters in A_1 permutes among themselves the letters of more than three transitive constituents of G_t. Now since the transitive constituents of G_t are of degree

* Note that Theorem 2 holds also for $t=5$.

† M. J. Weiss, these Transactions, vol. 30 (1928). See footnote on p. 337.

a power of two, this substitution fixes more than three letters of G_t. Hence it fixes t or more letters of G, and consequently G_t contains a substitution of order three, contrary to the present hypothesis.

If A_1 is of degree $t-2$ and transitive, H is of degree t^2-t and its substitutions record the permutations of the t^2-t sequences of two letters by the alternating group of degree t. Now a substitution of degree and order three of the alternating group of degree t fixes exactly $(t-3)(t-4)$ sequences. Hence a substitution of H_1 which is associated with a substitution of degree and order three of A_1 fixes exactly $(t-3)(t-4)$ transitive constituents of G_t. Then a substitution of I' which has a cycle of three letters in A_1 fixes at least $(t-3)(t-4)>2$ letters of G_t. Hence such a substitution fixes t or more letters of G.

Now A_1 cannot be transitive of degree $t-3$ or less, for then it is invariant in a group of at least three times its order and hence H_1 fixes three or more transitive constituents of G_t. Then in this case G_t also contains a substitution of order three. Thus A_1 is necessarily intransitive, containing an invariant alternating group of degree β, which fixes $t-\beta$ letters of A, and a constituent, transitive or intransitive, of degree $\leqq t-\beta$.

Moreover the constituent B is of degree $<t-2$. If B is of degree $t-2$, A_1 is of order $(t-2)!$. Then H is of degree $(t^2-t)/2$ and records the permutations of the $(t^2-t)/2$ transpositions of the alternating group of degree t. A substitution of degree and order three of A fixes exactly $(t-3)(t-4)/2$ transpositions. Hence the substitution of H associated with it fixes $(t-3)(t-4)/2>2$ transitive constituents of G_t. Thus unless B is of degree $<t-2$, G_t contains a substitution of order three.

It is necessary to determine the order of A_1. First it will be shown that A_1 contains all the substitutions of degree and order three of A which fix the letters of the constituent B. For this purpose it is necessary to have in mind Dyck's theorem* on transitive groups simply isomorphic to a given group. Now choose substitutions S_j not in A_1 such that A can be written in the array $A_1S_j, j=1, 2, \cdots, t!/(2m)$, where m is the order of A_1. Denote the substitutions of A_1 by σ_i, $i=1, 2, \cdots, m$. Multiply this array on the right by the substitutions σ_i. Then by Dyck's theorem, to each substitution σ_k of A_1, there corresponds the substitution

$$\begin{pmatrix} A_1S_j \\ A_1S_j\sigma_k \end{pmatrix} \qquad (j = 1,2, \cdots, t!/(2m)),$$

and the substitutions of H may be regarded as written on the sets of letters

* W. Dyck, Mathematische Annalen, vol. 22 (1883), pp. 70–108.

A_1S_j. Thus the substitution of H_1 associated with the substitution σ_k of A_1 is the above substitution.

Now suppose that S is a substitution of degree and order three of A which fixes the letters of the constituent B and which is not found in A_1. The sets A_1S and A_1S^2 found in the above array are distinct, for if these sets had a substitution in common, that is, if

$$\sigma_m S = \sigma_n S^2,$$

then

$$S = \sigma_n^{-1}\sigma_m,$$

but S is not a substitution of A_1. Further let σ be a substitution of degree and order three of A_1 on the letters of the constituent B only. Then σ and S are commutative since they are written on different sets of letters. Hence to the substitution σ corresponds the substitution

$$\begin{pmatrix} A_1S_j \\ A_1S_j\sigma \end{pmatrix} \qquad (j = 1, 2, \cdots, t!/(2m)),$$

which fixes three of the sets A_1S_j, namely, A_1S_1, A_1S, and A_1S^2, where $S_1=1$. Thus since the letters of H are also the transitive constituents of G_t, the substitution of order three of I' which coincides with σ in its cycle on the letters of A_1 fixes at least three transitive constituents of G_t, and hence is found in a G_t.

Thus A_1 also contains the alternating group on the $t-\beta$ letters not found in the constituent B. Hence the minimum order of A_1 is $\beta!(t-\beta)!/4$.

Now consider that subgroup D of A_1 which is the direct product of the alternating group of degree β and a substitution T of degree and order three on the letters of the constituent of degree $t-\beta$. Since A and H are simply isomorphic groups, H contains a subgroup J which is simply isomorphic to the group D. Further the subgroup of H_1 which is simply isomorphic to the alternating group of degree β displaces letters from every transitive constituent of H_1, for if it fixes the letters of one of the transitive constituents, a substitution of degree and order three of the alternating group of degree β fixes at least three transitive constituents of G_t. The three transitive constituents fixed would be the one that H_1 fixes and the ones which are the letters of the transitive constituent of H_1 in question. Hence this substitution would fix at least three letters of G_t and G_t would contain a substitution of order three. Thus since the alternating group of degree $\beta(\beta \geqq 5)$ is simple, every transitive constituent of H_1 has a subgroup simply isomorphic to it and associated with it. Again for the above reason the substitution T displaces letters of every transitive constituent of H_1. Hence every transitive constituent of H_1 has a subgroup simply isomorphic to D and associated with D.

Let J' be one of the transitive constituents of J simply isomorphic to D. Consider the subgroup D_1 of D which is simply isomorphic to the subgroup J_1' which fixes one letter of J'. It will be shown that D_1 cannot contain a substitution of degree and order three. First it cannot contain T, for T is invariant in D. Thus D_1 is invariant in a group of at least three times its order and consequently J_1' fixes at least three letters of J'. Then the substitution of J_1' associated with a substitution of degree and order three of D_1 fixes at least three transitive constituents of G_t, and G_t contains a substitution of order three.

Since D_1 contains no substitution of degree and order three a theorem by Bochert* may be applied to determine its maximum order. Its order thus $\leqq 3\beta!/[(\beta+1)/2]!$, where $[q]$ denotes the integral part of q. Then a transitive constituent of J is of degree $\frac{1}{2}[(\beta+1)/2]!$ at least.

The value of β must now be determined so that the degree of H can be evaluated. Under the present hypothesis, the order of $A_1 \leqq \beta!(t-\beta)!/2$, and hence the minimum degree of H is $t!/\{\beta!(t-\beta)!\}$. The maximum degree of H is found by determining the minimum order of A_1. The latter has been seen to be $\beta!(t-\beta)!/4$ and thus the maximum degree of H is $2t!/\{\beta!(t-\beta)!\}$. When $t>8$, H is simply transitive,† and J must contain at least two transitive constituents. If J and consequently H_1 contains only one transitive constituent, H_1 fixes exactly two letters of H, for it has been seen (§5, paragraph 3) that H_1 cannot fix so many as three letters. Then H is imprimitive with systems of two letters. Its group in the systems is then either a doubly transitive group simply isomorphic to the alternating group of degree t or the alternating group itself. The former case contradicts Maillet's theorems. In the latter case a substitution of degree and order three of A_1 has a substitution of degree and order three associated with it in the group in the systems of H. Consequently the substitution of H_1 associated with a substitution of degree and order three of A_1 fixes all except 6 letters of H. Then a substitution of G_t which has a cycle of three letters in A_1 fixes more than three transitive constituents of G_t. This has been shown previously to be impossible.

Now the least degree of J must always be less than the maximum degree of H. Hence

$$[(\beta + 1)/2]! < 2t!/\{\beta!(t - \beta)!\}, \quad t > 8. \tag{1}$$

Note that when $t=8$, β is determined, for $t/2<\beta\leqq t-3$. When $t>8$ and $\beta<t-3$, inequality (1) will be used to determine β.

Since the degree of each transitive constituent of G_t is at least two, G_t

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* A. Bochert, Mathematische Annalen, vol. 40 (1889), p. 584.

† E. Maillet, Journal de Mathématiques, (5), vol. 1 (1895), p. 5.

has at most $(n-t)/2$ transitive constituents. Now the maximum number of transitive constituents of G_t must exceed or equal the minimum degree of H, namely,

$$(n-t)/2 \geqq t!/\{\beta!(t-\beta)!\}. \tag{2}$$

It remains to determine a lower bound for $(n-t)/2$. The right hand member of inequality (2) decreases as β increases, $\beta > t/2$. Now it has been seen that β must be chosen so that inequality (1) holds. Hence if $\beta(<t-3)$ is chosen to be one unit less than the least value of β which satisfies the inequality

$$[(\beta+1)/2]! \geqq 2t!/\{\beta!(t-\beta)!\}, \quad t > 8, \tag{3}$$

inequality (2) holds for the value of β thus determined.

The above discussion may be summarized as follows:

If A_1 has an alternating constituent of degree $>t/2$, inequality (2) gives the relation between n and t, with $t/2 < \beta \leqq t-3$ or $\beta(<t-3)$ determined by inequality (3).

6. Now it may be assumed that the transitive constituent of degree $\beta(>t/2)$ of A_1 is not alternating. Let it be imprimitive. The largest imprimitive group of degree β is of order $2(\beta/2)!^2$ or $2\{(\beta-1)/2\}!^2$, according as β is even or odd. Hence the order of $A_1 \leqq [\beta/2]!^2\,(t-\beta)!$ and H is of degree $t!/\{2[\beta/2]!^2(t-\beta)!\}$ at least. Thus in this case the following inequality holds:

$$(n-t)/2 \geqq t!/\{2[\beta/2]!^2(t-\beta)!\}, \quad t/2 < \beta \leqq t. \tag{4}$$

Let the constituent of degree β be primitive. Since a non-alternating primitive group contains no substitution of degree and order three, the theorem of Bochert quoted above may be used to determine the order of the constituent of degree β. The order of $A_1 \leqq \beta!(t-\beta)!/[(\beta+1)/2]!$ and H is of degree $t![(\beta+1)/2]!/\{2\beta!(t-\beta)!\}$ at least. Thus if the constituent of degree β is primitive, the following inequality holds:

$$(n-t)/2 \geqq t![(\beta+1)/2]!/\{2\beta!(t-\beta)!\}, \quad t/2 < \beta \leqq t. \tag{5}$$

The minimum of each of the right hand members of inequalities (4) and (5) will now be determined. Denote them by $m_1(\beta)$ and $m_2(\beta)$, respectively. Recall Stirling's formula for $\log t!$, namely,

$$\log t! = (t+\tfrac{1}{2})\log t - t + \tfrac{1}{2}\log 2\pi + \theta/(12t), \quad 0 < \theta < 1.$$

Then

$$\log m_1(\beta) = \left(t + \frac{1}{2}\right)\log t + \frac{\theta}{12t} - \log 2 - \left(2\left[\frac{\beta}{2}\right] + 1\right)\log\left[\frac{\beta}{2}\right]$$

$$+ 2\left[\frac{\beta}{2}\right] - \log 2\pi - \frac{\theta_1}{6\left[\frac{\beta}{2}\right]} - \left(t - \beta + \frac{1}{2}\right)\log(t - \beta)$$

$$- \beta - \frac{\theta_2}{12(t-\beta)}, \quad \frac{t}{2} < \beta \leqq t - 1, \; 0 < \theta_i < 1.$$

The cases β even and β odd must be treated separately in order to apply Stirling's formula. However in both cases, the second derivative of log $m_1(\beta)$ is negative. Hence the minimum of log $m_1(\beta)$ occurs when β assumes its least even or odd value or when β assumes its greatest even or odd value, $[t/2]+1 \leqq \beta \leqq t-1$. Consequently the minimum of $m_1(\beta)$ occurs for some one of these values of β. It is found that if t is odd, the minimum occurs when $\beta = t-1$. If t is even, the minimum occurs when $\beta = t$. Thus the minimum of $m_1(\beta)$ is $t!/(2[t/2]!^2)$ and the inequality (4) becomes

$$(n - t)/2 \geqq t!/(2[t/2]!^2). \tag{6}$$

Apply Stirling's formula to log $m_2(\beta)$. Then

$$\log m_2(\beta) = \left(t + \frac{1}{2}\right)\log t - \left(\beta + \frac{1}{2}\right)\log \beta - \left(\left[\frac{\beta+1}{2}\right] + \frac{1}{2}\right)\log\left[\frac{\beta+1}{2}\right]$$

$$- \log 2 - \left(t - \beta + \frac{1}{2}\right)\log(t-\beta) - \left[\frac{\beta+1}{2}\right] + \frac{\theta}{12t} - \frac{\theta_1}{12\beta}$$

$$+ \frac{\theta_2}{12\left[\frac{\beta+1}{2}\right]} - \frac{\theta_3}{12(t-\beta)}, \quad \frac{t}{2} < \beta \leqq t - 1, \; 0 < \theta_i < 1.$$

The cases β even and β odd must again be considered separately. However the second derivative of log $m_2(\beta)$ is again negative in both cases. Hence the minimum of log $m_2(\beta)$ occurs for the extreme values of β, and consequently the minimum of $m_2(\beta)$ occurs for some one of these extreme values of β. Consider the values of β at the lower end of the interval. When β is even the least value of β is $(t+1)/2$, $(t+3)/2$, $(t+2)/2$, $(t+4)/2$, according as $t=2k+1$, $2k-1$, $2k$, or $4m$, k odd. When β is odd the least value of β is $(t+3)/2$, $(t+1)/2$, $(t+4)/2$, $(t+2)/2$, according as t is of one of the above forms. For all these values of β except for $\beta=(t+3)/2$ and $t=9$, $m_2(\beta) \geqq m_1(t)$. However if $t=9$, the minimum of $m_2(\beta)$ occurs for $\beta=9$. Hence inequality (5) need not be considered for values of β at the lower end of the β interval.

Then only the values of $m_2(\beta)$ for $\beta = t-2$, $t-1$, and t need be considered. Now $m_2(t)$ is less than $m_2(t-2)$ or $m_2(t-1)$. Thus inequality (5) may be replaced by the inequality

$$(n - t)/2 \geqq \tfrac{1}{2}[(t + 1)/2]!. \tag{7}$$

7. If no transitive constituent of A_1 is of degree $> t/2$, the order of $A_1 \leqq \frac{1}{2}[t/2]!^2$ and for this case

$$(n - t)/2 \geqq t!/[t/2]!^2. \tag{8}$$

However since the right hand member of inequality (8) is greater than the right hand member of inequality (6), the former may be discarded.

8. Thus if G_t is of order 2^m and $t \geqq 8$, the following theorem has been proved:

Theorem 3. *Let the subgroup that fixes t letters of a t-ply transitive group of degree n and class* >3 *be of order* 2^m. *Then if* $t \geqq 8$,

$$(n - t)/2 \geqq t!/\{\beta!(t - \beta)!\},$$

where β *is an integer chosen such that* $t/2 < \beta \leqq t-3$, *or chosen to be one unit less than the least value of* β *which satisfies the inequality*

$$[(\beta + 1)/2]! \geqq 2t!/\{\beta!(t - \beta)!\};$$

or

$$n - t \geqq [(t + 1)/2]!;$$

or

$$n - t \geqq t!/[t/2]!^2.$$

The symbol $[s]$ *denotes the integral part of s.*

When t is sufficiently great, it will be shown that the last inequality in Theorem 3 is the only one necessary to consider. Comparing the logarithms of the right hand members of the second and third inequalities, it is found by means of Stirling's formula, that when $t \geqq 16$, the former is always greater than the latter. Hence the former may be discarded when $t \geqq 16$.

Further analyze the inequality (3) which limits β, and write it in the form

$$V(\beta) = [(\beta + 1)/2]!\beta!(t - \beta)!/(2t!) \geqq 1.$$

Then if β is so chosen that $\log V(\beta) \geqq 0$, the inequality (3) holds. Now by Stirling's formula, it is found that $\log V(\beta)$ is positive for $\beta = t/2$ as soon as $t \geqq 160$. Then for $t \geqq 160$, β may be chosen equal to $t/2$, but $\beta > t/2$ by hypothesis. Hence when $t \geqq 160$ the first formula in Theorem 3 may also be discarded.

For later comparison purposes it is of interest to find the principal value of $\log(n-t)$ when t becomes infinite. Now

$$\log(n-t) \geqq \log t! - 2\log [t/2]!.$$

The right hand member of this inequality, when expanded by Stirling's formula, becomes

$$\begin{aligned}(t+\tfrac{1}{2})\log t - (t+1)\log(t/2)&\\ +\theta/(12t) - \theta_1/(3t) - \tfrac{1}{2}\log 2\pi&\\ &= (t+1)\log 2 - \tfrac{1}{2}\log t + \theta/(12t) - \theta_1/(3t) - \tfrac{1}{2}\log 2\pi,\end{aligned}$$

where $0<\theta_i<1$, $i=1, 2$, the principal value of which is $t \log 2$. Thus

$$\log(n-t) \geqq t!\log 2(1+\epsilon), \tag{9}$$

where ϵ approaches 0 as t approaches ∞.

G_t OF ORDER $p^\alpha m$, p AN ODD PRIME

9. In the remaining part of this paper it will be assumed that the order g_t of the subgroup G_t which fixes t letters of the t-ply transitive group G is divisible by an odd prime p. The analysis of this case is an extension of the method given by Jordan in his paper in Liouville's Journal of 1895.

Let p be the greatest prime which divides the order of G_t. The following slight extension of a theorem by Jordan* will be needed in the development of the theory for the case under discussion:

Consider a Sylow subgroup P of G_t of order p^α. Then $g_t = vp^\alpha(rp+1)$, where vp^α is the order of the largest subgroup of G_t in which P is invariant. Let $a_1, a_2, \cdots$ be the letters that G_t displaces and $b_1, b_2, \cdots, b_t$ the t letters that G_t fixes. Since G is t-ply transitive, there exists a substitution $S=(b_ib_k)(b_lb_m)(a\cdots)\cdots$ which transforms G_t into itself, b_i, b_k, b_l, b_m being any four of the above t letters. In the group of order $2g_t$ thus obtained, let the order of the largest group in which P is invariant be v_1p^α. Then $v_1=2v$ or v according as the new substitutions introduced do or do not transform P into itself. Sylow's theorem shows that $v_1=2v$, for $v_1p^\alpha(r_1p+1)=2vp^\alpha(rp+1)$, from which it is found that $v_1\equiv 2v$, mod p. Then for each set of possible values of the subscripts i, k, l, m there exists a substitution S_j which transforms P into itself. Now form the group $W=\{P, S_1, S_2, \cdots\}$. It contains P invariantly and also P's characteristic elementary subgroup L. Further W has an alternating constituent on the t letters that G_t fixes. The above discussion may be summarized as follows:

* C. Jordan, Journal de Mathématiques, (5), vol. 1 (1895), p. 37.

If the order of the subgroup which fixes t letters of a t-ply transitive group G is divisible by an odd prime p, a subgroup W of G can always be found which contains an invariant elementary subgroup L of order p^β on the letters of G_t and which has an alternating constituent on the t letters that G_t fixes.

The substitutions of W are of the form $ABPC$, where A denotes a substitution on the letters of the alternating constituent of degree t, B a substitution which permutes the transitive constituents of L, P the product of substitutions $P_1, P_2, \cdots$, each of which permutes among themselves the letters of a transitive constituent of L, and C a substitution on the letters of G_t not contained in L. The substitutions P_1, say, on the p^μ letters of a transitive constituent of L either do or do not generate a subgroup of the holomorph of the elementary group of degree p^μ which has a quotient group simply isomorphic to the alternating group of degree t. If the former be true, Jordan has shown that

$$\mu \geqq t - 3 - \log t/\log 2.*$$

Now the degree of a transitive constituent of L is $<n-t$. Hence

$$n - t > p^\mu,\ p > 2,\ \mu \geqq t - 3 - \log t/\log 2. \tag{10}$$

10. Suppose that this inequality does not hold. Then none of the sets of subsitutions P_i, $i=1, 2, \cdots$, generate a subgroup of the holomorph of the elementary group of degree p^{μ_i} which has a quotient group simply isomorphic to the alternating group of degree t. Hence W contains a subgroup $\overline{W}$, with an alternating constituent of degree t, in which each of the sets of substitutions P_i, except those forming the transitive constituents of L that are permuted in W, reduces to the identity. Now note that the elementary group L is of degree $\geqq(n-1)/2$, n being the degree of G, for the degree of L must be equal to or greater than the class of G. Then substitutions of $\overline{W}$ must permute some transitive constituents of L, for, otherwise, a substitution which has a cycle of three letters in the alternating group of degree t is of degree $\leqq 3+n-t-(n-1)/2<(n-1)/2$, the class of G.

Return to a consideration of the subgroup W. Since the substitutions of $\overline{W}$ permute some transitive constituents of L, the group W, which contains $\overline{W}$, also permutes some. The substitutions B together with those of L generate imprimitive groups whose systems of imprimitivity are the transitive constituents of L, and which are multiply isomorphic to the alternating group of degree t. Denote by $\mathcal{B}_1, \mathcal{B}_2, \cdots$, respectively, the groups in the systems of these imprimitive groups, and by $\mathcal{C}_1, \mathcal{C}_2, \cdots$ the groups generated by the substitutions C. Now the groups $\mathcal{B}_i$ and $\mathcal{C}_i$ are either all simply

* C. Jordan, Bulletin de la Société Mathématique de France, vol. 1 (1872), p. 55.

isomorphic to the alternating group of degree t or some are not. Assume the latter to be true.

The groups $\mathcal{B}_i$ and $\mathcal{C}_i$ which are multiply isomorphic to the alternating group of degree t then have invariant subgroups which fix the t letters of the alternating constituent. Denote these invariant subgroups by R. Now all the substitutions $S_j=(b_ib_k)(b_lb_m)(a\cdots)\cdots$ of W transform R into itself. Then unless the order of R is a power of two, choose its Sylow subgroup P_1 of order $p_1^{\alpha_1}$, p_1 an odd prime. The group P_1 reduces to R if R is of order a power of two. Then as in §9, paragraph 2, a group W_1 can be found in W which transforms P_1 into itself and which has an alternating constituent on the t letters that G_t fixes. The substitutions S_j, $j=1, 2, \cdots$, in this case are written on the letters of the alternating constituent of degree t, on the transitive constituents of L regarded as single symbols, and on the letters of G_t itself.

Consider the characteristic elementary subgroup L_1 of W_1. The degree of none of the transitive constituents of L_1 exceeds $(n-t)/p$ or $(n-2t+1)/2$, the former formula giving the maximum degree of an imprimitive group whose systems of imprimitivity are the transitive constituents of L, the latter formula giving the maximum degree of a transitive constituent of W which displaces none of the letters of L. Let Z be the greater of these two quantities. The substitutions of W_1 are of the form $A\overline{B}B'PP'C$, where $\overline{B}$ denotes substitutions which permute transitive constituents of L, but which with the substitutions of L generate imprimitive groups whose groups in the systems are simply isomorphic to the alternating group of degree t; B' denotes substitutions which permute the transitive constituents of L_1; P denotes the product of substitutions $P_1, P_2, \cdots$, each of which permutes among themselves the letters of a transitive constituent of L; P' denotes the product of substitutions $P_1', P_2', \cdots$, each of which permutes among themselves the letters of a transitive constituent of L_1; and C denotes substitutions on the letters of G_t not found in L or L_1. Recall that no one of the sets of substitutions P_i generates a group which has a quotient group simply isomorphic to the alternating group of degree t.

If one of the sets of substitutions P_i', P_1', say, on the p_1^γ letters of a transitive constituent of L_1, generates a subgroup of the holomorph of the elementary group of degree p_1^γ which has a quotient group simply isomorphic to the alternating group of degree t, Jordan's inequality gives

$$(11)\qquad Z \geqq p_1^\gamma,\ p_1 \geqq 2,\ \gamma \geqq t-3-\log t/\log 2.$$

If this inequality does not hold, it may be assumed that none of the above sets of substitutions P_i' generates a subgroup of the holomorph of an elementary group which has a quotient group simply isomorphic to the alternating group of degree t. Then W_1 contains a subgroup $\overline{W}_1$ which has an alternating constituent of degree t and in which each of the sets of substitutions P_i', except those forming the transitive constituents of L_1 that are permuted by W_1, reduces to the identity. If W_1 fixes all the transitive constituents of L_1, the substitutions of $\overline{W}_1$ reduce to $A\overline{B}C$. This case will be considered later. Then assume that the substitutions of W_1 permute some transitive constituents of L_1.

Consider the imprimitive groups generated by the substitutions B' and L_1. Denote their groups in the systems by $\mathcal{B}_1'$, $\mathcal{B}_2'$, $\cdots$, respectively, and the groups generated by the substitutions C, by $\mathcal{C}_1$, $\mathcal{C}_2$, $\cdots$, respectively. Again, unless all the groups $\mathcal{B}_i'$ and $\mathcal{C}_i$ are simply isomorphic to the alternating group of degree t, choose their invariant subgroups denoted by R_1 which fix the t letters of the alternating group and repeat the above analysis, obtaining a group W_2 which contains an invariant elementary subgroup L_2. If all the groups $\mathcal{B}_i'$ and $\mathcal{C}_i$ are simply isomorphic to the alternating group of degree t, the analysis has been completed for the present purpose. The substitutions of W_1 then have the form $A\overline{B}\overline{B}'\overline{C}PP'$, where A, $\overline{B}$, P, P' are defined as before; $\overline{B}'$ denotes substitutions which with those of L_1 generate imprimitive groups whose groups in the systems are simply isomorphic to the alternating group of degree t; and $\overline{C}$ denotes substitutions on the letters of G_t not found in L or L_1 which generate subgroups simply isomorphic to the alternating group of degree t. Note that none of the substitutions P_i or P_i' generate groups which have quotient groups simply isomorphic to the alternating group of degree t.

Now return to the case in which W_1 fixes all the transitive constituents of L_1, and consider the group $\overline{W}_1$, whose substitutions are of the form $A\overline{B}C$. If the substitutions C generate groups which are simply isomorphic to the alternating group of degree t, the analysis has been completed. If they do not, take their invariant subgroups denoted by $\overline{R}_1$, which fix the t letters of the alternating group, and apply the reasoning of the previous paragraphs, again obtaining a group $\overline{W}_2$ which contains an invariant elementary subgroup L_2.

Continue with this analysis until either a group W_j or $\overline{W}_j$ of the desired type is obtained, or an inequality, such as inequalities (10) or (11), derived from this analysis holds. Obviously these two inequalities are the most unfavorable of any which might be obtained in carrying this analysis further. Consequently any further inequalities may be neglected. Then it remains to investigate the case when a group W_j or $\overline{W}_j$ is obtained in which the substitutions are of the form

$$A\overline{B}\overline{B}'\overline{B}'' \cdots \overline{C}PP'P'' \cdots ,$$

where the substitutions $\overline{B}$, $\overline{B}'$, $\overline{B}''$, $\cdots$ denote substitutions which permute transitive constituents of L, L_1, L_2, $\cdots$, respectively, but which with the substitutions of the latter generate imprimitive groups whose groups in the systems are simply isomorphic to the alternating group of degree t; $\overline{C}$ denotes substitutions on the letters of G_t not found in L, L_1, L_2, $\cdots$, which generate groups simply isomorphic to the alternating group of degree t; $P, P', P'', \cdots$ denote the product of substitutions $P_1, P_2, \cdots, P_1', P_2', \cdots, P_1'', P_2'', \cdots$, each of which permute the letters of a transitive constituent of L, L_1, L_2, $\cdots$, respectively, among themselves, but which do not generate groups which have quotient groups simply isomorphic to the alternating group of degree t.

11. Thus suppose that W_j is the first group in the series W_i, $\overline{W}_i$, $i=1, 2, \cdots$, of the desired type. Let the groups in the systems of the imprimitive groups generated by $\overline{B}^m$ and L_m be denoted by $\mathcal{B}_v^m$, $v=1, 2, \cdots$, and $m=0, 1, \cdots$, $\mathcal{B}_v^0=\mathcal{B}_v$, and the groups generated by the substitutions $\overline{C}$, by $\mathcal{C}_v$. Recall that the groups $\mathcal{B}_v^m$ and $\mathcal{C}_v$ are all simply isomorphic to the alternating group of degree t. Consider a subgroup K of W_j which has an alternating constituent on the k letters $b_1, b_2, \cdots, b_k$, where $5 \leqq k \leqq t$. First suppose that K fixes more than one half the letters in each of the groups $\mathcal{B}_v^m$ and $\mathcal{C}_v$. Now all these letters or symbols fixed cannot reduce to the identity in G_t, for then the degree of a substitution of W_j which has a cycle of three letters in the alternating constituent of degree k is of degree $\leqq 3+(n-t)/2<(n-1)/2$. Then according to the theorem by Manning quoted in §3, G is alternating. Therefore corresponding to at least one symbol fixed and arising from the letters of L, L_1, or L_2, $\cdots$, there exists a group simply or multiply isomorphic to the alternating constituent of degree k which transforms an elementary group into itself. Now the degree of this group cannot exceed the degree of one of the systems of imprimitivity of an imprimitive group generated by the substitutions $\overline{B}^m$ and those of L_m. It has been shown that such a group permutes its systems according to a group simply isomorphic to the alternating group of degree t. Hence it has at least t systems and the degree of a system is at most $(n-t)/t$. If this imprimitive group is a group generated by the substitutions $\overline{B}$ and those of L, Jordan's inequality gives

$$(12) \qquad (n-t)/t \geqq p^\delta,\ p>2,\ 5 \leqq k \leqq t-1,\ \delta \geqq k-3-\log k/\log 2.$$

If the substitutions $\overline{B}$ reduce to the identity in K, let the group under discussion be a group generated bythe substitutions $\overline{B}'$ and those of L_1. The degree of this imprimitive group is then at most Z. Hence the degree of one of its systems is at most Z/t. In this case Jordan's inequality gives

(13) $Z/t \geqq p_1^{\delta_1},\ p_1 \geqq 2,\ 5 \leqq k \leqq t-1,\ \delta_1 \geqq k-3-\log k/\log 2.$

If the substitutions $\overline{B}'$ reduce to the identity in K, let the group in question be a group generated by the substitutions $\overline{B}''$ and those of L_2. Any further inequalities thus obtained evidently give more favorable bounds for the transitivity of G and hence may be discarded.

12. It may now be assumed that the group K displaces a half or more than a half of the letters in at least one of the groups simply isomorphic to the alternating group of degree t. Let Q be the particular group in which K displaces a half or more than one half of the letters. Let Γ_1 be the subgroup of the alternating constituent Γ of degree t which corresponds to the subgroup Q_1 that fixes one letter of Q. Now the degree of Q is at most $(n-t)/p$ or $(n-2t+1)/2$, according as Q has arisen from the letters of L or from the letters of G_t not found in L. On the other hand, the degree of Q is $t!/(2q)$ where q is the order of the subgroup Γ_1. Hence for this case the inequality

$$Z \geqq t!/(2q)$$

must be studied.

Let $\gamma_1, \gamma_2, \cdots$ be the degrees of the transitive constituents of Γ_1. Then the order of each transitive constituent group of Γ_1 divides $\gamma_i!$, respectively. Hence the order q divides $\gamma_1!\gamma_2! \cdots$.

Recall the present hypothesis that the group K fixes at most one half the letters in the group Q. Let Q_1 have v conjugates under Q. Then Γ_1 has v conjugates under Γ. Now those conjugates of Q_1 which fix letters that K fixes, but which fix no letters that K displaces, are invariant under K. Hence at most one half of the conjugates of Q_1 are invariant under K and consequently at most one half the conjugates of Γ_1 are invariant under the alternating group of degree k on the letters $b_1, b_2, \cdots, b_k$. Using this hypothesis Jordan* has shown that if one of the numbers γ_1 is greater than the number δ, defined to be the greatest integer less than the quantity $t-(t-k+1)\log 2/(k+\log 2)$, the corresponding transitive constituent group of Γ_1 cannot contain the alternating group of the same degree.

Thus there are the following three cases to consider:

I. All the numbers $\gamma_i \leqq \delta$.

II. One number $\gamma_1 > \delta$, and the corresponding transitive constituent of degree γ_1 of Γ_1 is imprimitive.

III. One number $\gamma_1 > \delta$, and the corresponding transitive constituent of degree γ_1 of Γ_1 is primitive.

In Case I, the order $q \leqq \frac{1}{2}\delta!(t-\delta)!$. In Case II recall that the largest imprimitive group of degree γ_1 is of order $2(\gamma_1/2)!^2$ or $2((\gamma_1-1)/2)!^2$ ac-

* C. Jordan, Journal de Mathématiques, (5), vol. 1 (1895), p. 51.

cording as γ_1 is even or odd. Thus $q \leqq [\gamma_1/2]!^2(t-\gamma_1)!$ where $[s]$ denotes the integral part of s. In Case III, Bochert's theorem may be used to determine the order of the primitive constituent, and hence $q \leqq (t-\gamma_1)!\gamma_1!/[(\gamma_1+1)/2]!$. These three cases thus lead to the following inequalities:

$$Z \geqq t!/\{\delta!(t-\delta)!\}, \tag{14}$$

$$Z \geqq t!/\{2[\gamma_1/2]!^2(t-\gamma_1)!\}, \tag{15}$$

$$Z \geqq t![(\gamma_1+1)/2]!/\{2\gamma_1!(t-\gamma_1)!\}. \tag{16}$$

It has been seen (§6) that the minimum of the right hand member of the inequality (15) occurs when $\gamma_1 = t$, and that inequality (16) need be considered only for $\gamma_1 = t$. Further Z may be taken equal to $(n-2t+1)/2$, for the greatest value of $(n-t)/p$ occurs when $p=3$, and the former number is always greater than the latter as soon as $n \geqq 4t-3$. Now $n \geqq 4t-3$, for the right hand members of the above inequalities are always $\geqq t$. Thus inequalities (14), (15), and (16) may be replaced by

$$(n-2t+1)/2 \geqq t!/\{\delta!(t-\delta)!\}, \tag{17}$$

$$(n-2t+1)/2 \geqq t!/\{2[t/2]!^2\}, \tag{18}$$

$$(n-2t+1)/2 \geqq \tfrac{1}{2}[(t+1)/2]!. \tag{19}$$

In the list of inequalities in §§9, 10, and 11 substitute the least possible values for p, p_1, and Z. Then only the following inequalities need be considered:

$$n - t \geqq 3^\alpha, \quad \alpha \geqq t - 3 - \log t/\log 2, \tag{20}$$

$$(n-2t+1)/2 \geqq 2^\alpha, \tag{21}$$

$$(n-t)/t \geqq 3^\beta, \quad \beta \geqq k - 3 - \log k/\log 2, \quad 5 \leqq k \leqq t-1, \tag{22}$$

$$(n-2t+1)/(2t) \geqq 2^\beta. \tag{23}$$

In inequality (20), $\alpha > 3$, for the holomorph of an elementary group of degree 9 or 27 obviously has no quotient group simply isomorphic to the alternating group of degree $t(\geqq 8)$. Likewise in inequality (21), $\alpha > 3$. For a similar reason $\beta > 3$ in inequalities (22) and (23), for in these cases the alternating group of degree t is replaced by one of degree k. Although these same restrictions on α and β might not hold if larger primes had been substituted for p and p_1, similar considerations show that greater values for p and p_1 give still more favorable values for n.

Now write inequalities (22) and (23) in the forms $n \geqq 3^\beta t + t$ and $n \geqq 2^{\beta+1}t + 2t - 1$, respectively. Since $3^\beta \geqq 2^{\beta+1} + 1 - 1/t$ for all values of $\beta > 3$, inequality (22) may be discarded. Again write inequalities (20) and (21) in the forms $n \geqq 3^\alpha + t$ and $n \geqq 2^{\alpha+1}+2t-1$, respectively. Then if $2^{\alpha+1}+t-1<3^\alpha$, inequality (20) may be discarded. Now $t-1<2^\alpha$, $\alpha>3$. Since $2^\alpha<3^{\alpha-1}$ for all values of $\alpha>3$, the above inequality holds.

Further, compare the right hand members of inequalities (18) and (19) with the right hand member of inequality (21). The logarithm of the right hand member of inequality (18) is always greater than the logarithm of the right hand member of inequality (21). Hence inequality (18) may be discarded. Again the logarithm of the right hand member of inequality (19) is greater than the logarithm of the right hand member of inequality (21) except when $t=8$, $\alpha=4$. However, when $t=8$, inequality (17) gives a still lower bound for n than inequality (19). Thus inequality (19) may be discarded for all values of t.

13. The results of the study of the t-ply transitive group G, the order of whose subgroup that fixes t letters is divisible by an odd prime, may be summarized in the following theorem:

THEOREM 4. *Let the order of the subgroup that fixes t letters of a t-ply transitive group of degree n and class* >3 *be divisible by an odd prime. Then if* $t \geqq 8$,

$$(n - 2t + 1)/2 \geqq 2^\alpha,\ \alpha \text{ an integer} > 3 \text{ and} \geqq t - 3 - \log t/\log 2;$$

or

$$(n - 2t + 1)/2 \geqq 2^\gamma t,\ \gamma \text{ an integer} > 3 \text{ and} \geqq k - 3 - \log k/\log 2,$$

where k is an integer such that $5 \leqq k \leqq t-1$; *or*

$$(n - 2t + 1)/2 \geqq t!/\{\delta!(t - \delta)!\},$$

where δ *is the greatest integer less than the quantity* $t-(t-k+1)\log 2/(k+\log 2)$.

It may be of interest to find the principal value, when t becomes infinite, of $(n-2t+1)/2$, or, which is more convenient, of $\log((n-2t+1)/2)$. The logarithm of the right hand member of the first inequality in Theorem 4 obviously has the principal value $t \log 2$. It remains to find the principal values of the right hand members of the second and third inequalities in the above theorem. Denote the logarithm of the right hand member of the second inequality by $f(k)$ and that of the third by $g(k)$. Thus

$$f(k) = (k - 3)\log 2 - \log k + \log t,$$

and by Stirling's formula

$$g(k) = (t + \tfrac{1}{2})\log t - (\delta + \tfrac{1}{2})\log \delta - (t - \delta + \tfrac{1}{2})\log (t - \delta) - \tfrac{1}{2}\log 2\pi$$
$$+ \theta/(12t) - \theta_1/(12\delta) - \theta_2/(12(t - \delta)), \text{ where } 0 < \theta_i < 1,\ i = 0,1,2.$$

Now $f(k)$ is a function which increases in value with k, while $g(k)$ is a function which decreases in value as k increases. First determine k so that the principal values of these functions are equal. Assume k and t of equal order. Then the principal value of $t-\delta$ is $\{(t-k)\log 2\}/k$. Now in $g(k)$ replace $\log \delta$ by

$$\log t + \log \{1 - (t-\delta)/t\} = \log t - (t-\delta)/t - (t-\delta)^2/(2t^2) - \cdots .$$

Then the principal value of $g(k)$ is $\{(t-k)\log 2 \log t\}/k$, while the principal value of $f(k)$ is obviously $k \log 2$. Setting these two values equal, it is found that

$$k = -\tfrac{1}{2}\log t + \tfrac{1}{2}(4t\log t + \log^2 t)^{1/2},$$

of which the principal value is $(t \log t)^{1/2}$. Thus k is of order less than the order of t, and using this value of k, the principal value of $\log((n-2t+1)/2)$, derived from the second and third inequalities in the above theorem, is $(t\log t)^{1/2}\log 2$. This result is evidently less favorable than the one derived from the first inequality, and hence for the case G_t of order $p^\beta m$, p an odd prime, the following relation holds:

$$\log((n-2t+1)/2) \geqq (t\log t)^{1/2}\log 2(1-\eta), \tag{24}$$

where η approaches 0 as t approaches ∞.

In §8, it was shown that for the case G_t of order 2^v the principal value of $\log(n-t)$ is $t \log 2$. Evidently this result gives a more favorable relation between n and t than the one just established. Hence inequality (24) holds for all t-ply transitive groups of class >3. This inequality is similar to the one found by Jordan in his 1895 paper, namely,

$$\log(n-t) \geqq (t\log t)^{1/2}\log 2(1-\epsilon),$$

where ϵ approaches 0 as t approaches ∞.

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